Eur. Phys. J. C 49, 721–730 (2007)
DOI 10.1140/epjc/s10052-006-0138-x

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

On Khalfin's improvement of the LOY effective Hamiltonian for neutral meson complex

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Received: 19 March 2006 / Revised version: 16 August 2006 /
Published online: 21 November 2006 –

Abstract. The general properties of the effective Hamiltonian for the neutral meson system improved by Khalfin in 1980 are studied. It is shown that contrary to the standard result of the Lee–Oehme–Yang (LOY) theory, the diagonal matrix elements of this effective Hamiltonian cannot be equal in a CPT invariant system. It is also shown that the scalar product of short, $|K_{\rm S}\rangle$, and long, $|K_{\rm L}\rangle$, living superpositions of neutral kaons cannot be real when CPT symmetry is conserved in the system under consideration, whereas within the LOY theory such a scalar product is real.

PACS. 03.65.Ca; 11.10.St; 11.30.Er; 13.20.Eb; 13.25.Es

1 Introduction

A formalism convenient for the description of the properties of neutral K mesons and their time evolution was proposed by Lee, Oehme and Yang (LOY) in 1956 [1]. Within the LOY approach the Weisskopf–Wigner (WW) approximation used for studying the time evolution of a single quasistationary state [2] was adapted to the case of two state (two particle) subsystems. Within the WW and LOY approaches it is assumed that the time evolution of the total system under consideration containing one or two quasistationary states is governed by the Schrödinger equation

$$\mathrm{i}\frac{\partial|\psi(t)\rangle}{\partial t} = H|\psi(t)\rangle, |\psi(t=0)\rangle = |\psi_0\rangle\,, \qquad (1)$$

where H is the self-adjoint Hamiltonian of the total system and $|\psi(t)\rangle$ and $|\psi_0\rangle$ are vectors belonging to the total state space $\mathfrak{H}$ of the system. Using the interaction representation LOY found approximate solutions of the time dependent equation (1) for the two state problem and then the Schrödinger-like equation with nonhermitian effective Hamiltonian $\mathcal{H}_{||}$ governing the time evolution of these two states [1, 3]. Unfortunately not all steps of the approximations applied in [1, 3] to obtain the solution of the problem are well defined. Some attempts to give a more exact derivation of the equation governing the time evolution of the two state subsystem and to improve the LOY result were based on the approach exploited in [4] for studying the properties of unstable states. As an example of such attempts one can consider the method described in [5, 6]. This method reproduces not only the LOY effective Hamiltonian $\mathcal{H}_{||}$, but it also enables one to relatively simply improve the LOY effective Hamiltonian. It appears that an examination of the formulae for the matrix elements of the improved $\mathcal{H}_{||}$ obtained in [6] suggests that those formulae contain some inconsistencies. The Khalfin method to improve the effective Hamiltonians for multi-component systems is elegant and seems to be important as it has the potential of correctly describing the process in question. This is why we decided to use it to find the exact expressions for the matrix elements of Khalfin's effective Hamiltonian. The aim of this paper is to give a detailed analysis of the approximation described and exploited in [5, 6] and to compare the properties of the effective Hamiltonians obtained within this approximation with other effective Hamiltonians. In Sect. 2 we review briefly the method used in [5] and in the first two sections of [6] to obtain $\mathcal{H}_{||}$. In Sect. 3 we describe an improved effective Hamiltonian for the neutral K complex derived by Khalfin in Sect. 3 of his paper [6], and we give the formulae for the matrix elements of this Hamiltonian free from the above mentioned inconsistencies. In Sect. 4 we study and discuss some properties of the mentioned improved effective Hamiltonian not considered in [6]. Section 5 contains a short discussion of the properties of Khalfin's improved effective Hamiltonian as well as our final remarks.

2 Neutral kaons within Weisskopf–Wigner approximation

Let us follow [5, 6] and denote by H a self-adjoint Hamiltonian for the total physical system containing a neutral

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meson subsystem and assume that

$$H = H^0 + H^{\mathrm{w}}, \tag{2}$$

where H^0 denotes the sum of strong and electromagnetic interactions and H^{w} stands for weak interactions. Next it is assumed that the operator H^0 has a complete set of eigenvectors $\{|K^0\rangle, |\bar{K}^0\rangle, |F\rangle\}$, where

$$\begin{aligned} H^0|K^0\rangle = m_{K^0}|K^0\rangle\,, \quad H^0|\bar{K}^0\rangle = m_{\bar{K}^0}|\bar{K}^0\rangle\,, \\ H^0|F\rangle = E_F|F\rangle\,, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \langle K^0|\bar{K}^0\rangle = 0\,, \quad \langle K^0|K^0\rangle = \langle \bar{K}^0|\bar{K}^0\rangle = 1\,, \\ \langle K^0|F\rangle = \langle \bar{K}^0|F\rangle = 0\,. \end{aligned}$$

The vectors $|K^0\rangle$ and $\bar{K}^0\rangle$ are identified with the state vectors of the neutral K and anti-K mesons. The vectors $|F\rangle$ correspond to decay products of the neutral kaons.

Usually it is assumed that the strong and electromagnetic interactions preserve the strangeness S, i.e. that

$$[H^0, S] = 0\,, \tag{4}$$

and the same assumption is used in [5, 6]. The vectors $|K^0\rangle$ and $\bar{K}^0\rangle$ are the eigenvectors of the operator S:

$$S|K^0\rangle = (+1)|K^0\rangle, \quad S|\bar{K}^0\rangle = (-1)|\bar{K}^0\rangle\,. \tag{5}$$

In [5, 6] it is also assumed that the strong and electromagnetic interactions are CPT invariant:

$$[H^0, \mathcal{CPT}] = 0\,, \tag{6}$$

and that they preserve CP symmetry

$$[H^0, \mathcal{CP}] = 0\,. \tag{7}$$

Here $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ denote operators realizing charge conjugation, parity and time reversal respectively, for vectors in $\mathfrak{H}$.

The following phase convention is used in [5, 6]:

$$\mathcal{CPT}|\bar{K}^0\rangle = |K^0\rangle\,. \tag{8}$$

This last relation and CPT invariance of H^0 (6) imply that

$$m_{K^0} = \langle K^0|H^0|K^0\rangle = m_{\bar{K}^0} = \langle \bar{K}^0|H^0|\bar{K}^0\rangle = m. \tag{9}$$

The authors of [5, 6] deriving the effective Hamiltonian governing the time evolution in the subspace of states spanned by the vectors $|K^0\rangle, |\bar{K}^0\rangle$ start from the solution of the Schrödinger equation (1) for $t > t_0 = 0$ having the following form:

$$|\psi(t)\rangle = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}Et} G_+(E)\mathrm{d}E|\psi_0\rangle, \quad (t > 0), \tag{10}$$

where

$$G_+(E) = (E - H + \mathrm{i}\varepsilon)^{-1}. \tag{11}$$

Using (2) one finds [4]

$$G_+(E) \equiv (E - H^0 + \mathrm{i}\varepsilon)^{-1} + (E - H^0 + \mathrm{i}\varepsilon)^{-1} H^{\mathrm{w}} G_+(E). \tag{12}$$

The solutions $|\psi(t)\rangle$ of (1) can be expanded into a complete set of eigenvectors $\{|K^0\rangle, |\bar{K}^0\rangle, |F\rangle\}$ of the operator H^0. For the problem considered it is assumed that one has

$$|\psi_0\rangle = a_{K^0}(0)|K^0\rangle + a_{\bar{K}^0}(0)|\bar{K}^0\rangle \tag{13}$$

at the initial instant $t = t_0 = 0$. So there are no decay products in the system at $t_0 = 0$. They can be detected there only at the instant $t > t_0 = 0$. Thus

$$\begin{aligned} |\psi(t)\rangle &= a_{K^0}(t)|K^0\rangle + a_{\bar{K}^0}(t)|\bar{K}^0\rangle + \sum_F \sigma_F(t)|F\rangle \\ &\stackrel{\text{def}}{=} |\psi(t)\rangle_{\parallel} + |\psi(t)\rangle_{\perp} \end{aligned} \tag{14}$$

at $t > t_0 = 0$ and therefore $\sigma_F(0) = 0$ and $\sigma_F(t) \neq 0$ for $t > 0$. Here

$$\begin{aligned} |\psi(t)\rangle_{\parallel} &= a_{K^0}(t)|K^0\rangle + a_{\bar{K}^0}(t)|\bar{K}^0\rangle \quad \text{and} \\ |\psi(t)\rangle_{\perp} &= \sum_F \sigma_F(t)|F\rangle \end{aligned}$$

and $|\psi(t)\rangle_{\parallel} \in \mathfrak{H}_{||} \subset \mathfrak{H}$, $|\psi(t)\rangle_{\perp} \in \mathfrak{H} \ominus \mathfrak{H}_{||}$. The subspace $\mathfrak{H}_{||}$ is a two dimensional subspace of the state space $\mathfrak{H}$ spanned by the orthogonal vectors $|K_0\rangle, |\bar{K}_0\rangle$.

We have

$$\begin{aligned} a_{K^0}(t) = \langle K^0|\psi(t)\rangle \equiv \langle K^0|\psi(t)\rangle_{\parallel}, \\ a_{\bar{K}^0}(t) = \langle \bar{K}^0|\psi(t)\rangle \equiv \langle \bar{K}^0|\psi(t)\rangle_{\parallel}. \end{aligned} \tag{15}$$

From (10) one infers that

$$\begin{aligned} a_\alpha(t) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}Et} \sum_\beta \langle \alpha|G_+(E)|\beta\rangle a_\beta(0)\mathrm{d}E, \\ (t > 0), \end{aligned} \tag{16}$$

where $\alpha, \beta = K^0, \bar{K}^0$. Using the relation (12) and condition (9), after some algebra one can rewrite the solutions (16) of (1) in matrix form as follows:

$$a(t) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}Et} \frac{\mathrm{d}E}{E - m - R(E) + \mathrm{i}\varepsilon} a(0), \tag{17}$$

where $t > 0$, and $a(t)$ and $a(0)$ are one column matrices

$$a(t) = \begin{pmatrix} a_{K^0}(t) \\ a_{\bar{K}^0}(t) \end{pmatrix}, \quad a(0) = \begin{pmatrix} a_{K^0}(0) \\ a_{\bar{K}^0}(0) \end{pmatrix}, \tag{18}$$

and $R(E)$ is the (2×2) matrix with matrix elements $R_{\alpha\beta}(E)$, where $\alpha, \beta = K^0, \bar{K}^0$ and

$$R_{\alpha\beta}(E) = \langle\alpha|H^{\rm w}|\beta\rangle + \sum_F \langle\alpha|H^{\rm w}|F\rangle \frac{1}{E - E_F + {\rm i}\varepsilon}\langle F|H^{\rm w}|\beta\rangle + \sum_F\sum_{F'}\Big\{\langle\alpha|H^{\rm w}|F\rangle\frac{1}{E-E_F+{\rm i}\varepsilon} \times \langle F|H^{\rm w}|F'\rangle \frac{1}{E-E_{F'}+{\rm i}\varepsilon}\langle F'|H^{\rm w}|\beta\rangle\Big\} + \ldots \tag{19}$$

$$\equiv \langle\alpha|H^{\rm w}|\beta\rangle - \Sigma_{\alpha\beta}(E)\,, \tag{20}$$

and

$$\Sigma_{\alpha\beta}(E) = \sum_{F,F'}\langle\alpha|H^{\rm w}|F\rangle\left\langle F\left|\frac{1}{QHQ-E-{\rm i}0}\right|F'\right\rangle\langle F'|H^{\rm w}|\beta\rangle \equiv \left\langle\alpha\left|H^{\rm w}Q\frac{1}{QHQ-E-{\rm i}0}QH^{\rm w}\right|\beta\right\rangle \tag{21}$$

$$\equiv \langle\alpha|\Sigma(E)|\beta\rangle\,, \tag{22}$$

and

$$\Sigma(E) = PHQ\frac{1}{QHQ-E-{\rm i}0}QHP\,, \tag{23}$$

$$Q = \sum_F |F\rangle\langle F|\,, \tag{24}$$

$$P = \mathbb{I} - Q \equiv |K^0\rangle\langle K^0| + |\bar{K}^0\rangle\langle\bar{K}^0|\,. \tag{25}$$

The relation (17) is the exact formula for the solutions of (1) for $t > 0$. The problem is how to evaluate the integral (17) and thus the amplitudes $a_\alpha(t)$. Usually this is possible within the use of some approximate methods. Depending on the methods used one obtains more or less accurate expressions for $a_\alpha(t)$ and therefore a more or less accurate description of the properties of the physical system considered.

From the experimental data it is known that

$$\langle K^0|H^0|K^0\rangle = m \gg \langle K^0|H^{\rm w}|K^0\rangle = \Delta_w m. \tag{26}$$

This enables one to assume that $|R_{\alpha\beta}(E)| \ll m$. So the conclusion that the position of the pole of the expression under the integral in (17) is very close to m seems to be reasonable. Thus one can expect that replacing $R(E)$ by $R(m)$ in (17) should not cause a large deviation from the exact value of this integral. Making use of this conclusion the value of the integral (17) can be computed within the so-called Weisskopf–Wigner approximation [2, 5, 6].

According to [6] the WW approximation consists of the following.

1. Taking into account only the pole contribution into the value of the integral (17) (i.e., a neglecting all the cut and threshold, contributions etc. to the value of the integral (17)).
2. Replacing $R(E)$ by its value for $E = m$, (i.e. inserting $R(m)$ instead of $R(E)$ in (17)).

(Note that it is rather difficult to find an exact estimation of the error generated by such a procedure). Applying this prescription to the integral (17) yields

$$a(t) \simeq a^{\rm WW}(t) = -\frac{1}{2\pi{\rm i}}\int_{-\infty}^{\infty}{\rm e}^{-{\rm i}Et}\frac{{\rm d}E}{E-m-R(m)+i\varepsilon}a(0) \tag{27}$$

$$\equiv {\rm e}^{-{\rm i}\mathcal{H}^{\rm WW}t}a(0) \tag{28}$$

for $t > 0$, where

$$\mathcal{H}^{\rm WW} \stackrel{\rm def}{=} m\mathbb{I}_{\|} + R(m) \equiv M^{\rm WW} - \frac{\rm i}{2}\Gamma^{\rm WW} \tag{29}$$

is an nonhermitian operator, $M^{\rm WW} = (M^{\rm WW})^+, \Gamma^{\rm WW} = (\Gamma^{\rm WW})^+$, and

$$\mathbb{I}_{\|} = \begin{pmatrix} 1 & , & 0 \\ 0 & , & 1\end{pmatrix}. \tag{30}$$

Matrix elements of $\mathcal{H}^{\rm WW}$ have the following form:

$$\mathcal{H}^{\rm WW}_{\alpha\beta} = m\delta_{\alpha\beta} + R_{\alpha\beta}(m)\,, \tag{31}$$

$$\equiv m\delta_{\alpha\beta} + H^{\rm w}_{\alpha\beta} - \Sigma_{\alpha\beta}(m)\,, \tag{32}$$

where $\alpha, \beta = K^0, \bar{K}^0$ and $H^{\rm w}_{\alpha\beta} = \langle\alpha|H^{\rm w}|\beta\rangle$.

All operators appearing in the definition (29) act in the two dimensional subspace of states $\mathfrak{H}_{||}$. From (28) it follows that $a(t) \simeq a^{\rm WW}(t) \in \mathfrak{H}_{||}$ solves the following Schrödinger-like equation:

$${\rm i}\frac{\partial a^{\rm WW}(t)}{\partial t} = \mathcal{H}^{\rm WW}a^{\rm WW}(t)\,, \tag{33}$$

where the operator $\mathcal{H}^{\rm WW}$ is the effective Hamiltonian for vectors belonging to the subspace $\mathfrak{H}_{||}$. Matrix elements of $\mathcal{H}^{\rm WW}$ are defined by the formulae (20). They are exactly the same as those obtained by LOY [1, 3, 7–13].

If the assumption (6) is completed by the following one:

$$[H, CPT] = 0\,, \tag{34}$$

that is, if one assumes that the system containing neutral kaons is CPT invariant, then using the relations (8) and (9) one easily finds from (20) that

$$R_{K^0K^0}(m) = R_{\bar{K}^0\bar{K}^0}(m)\,, \tag{35}$$

and thus

$$\mathcal{H}^{\rm WW}_{K^0K^0} = \mathcal{H}^{\rm WW}_{\bar{K}^0\bar{K}^0}\,, \quad M^{\rm WW}_{K^0K^0} = M^{\rm WW}_{\bar{K}^0\bar{K}^0}\,, \quad \Gamma^{\rm WW}_{K^0K^0} = \Gamma^{\rm WW}_{\bar{K}^0\bar{K}^0}\,. \tag{36}$$

These relations are the standard conclusions of the LOY theory for CPT invariant physical systems [3, 5, 7–18].

Assuming that the interactions $H^{\rm w}$ responsible for the decay processes in the system considered violate CP symmetry, $[\mathcal{CP}, H] \neq 0$, one finds another important result of the LOY approach:

$$R_{K^0\bar{K}^0}(m) \neq R_{\bar{K}^0K^0}(m)\,, \tag{37}$$

which implies that

$$\mathcal{H}^{\mathrm{WW}}_{K^0\bar{K}^0} \neq \mathcal{H}^{\mathrm{WW}}_{\bar{K}^0K^0}\,, \quad M^{\mathrm{WW}}_{K^0\bar{K}^0} \neq M^{\mathrm{WW}}_{\bar{K}^0K^0}\,, \\ \Gamma^{\mathrm{WW}}_{K^0\bar{K}^0} \neq \Gamma^{\mathrm{WW}}_{\bar{K}^0K^0}\,. \tag{38}$$

The eigenvalue equations for $\mathcal{H}^{\mathrm{WW}}$ have the following form [5,6]:

$$\mathcal{H}^{\mathrm{WW}} a^{\mathrm{WW}}_{\mathrm{S}} = \lambda_{\mathrm{S}} a^{\mathrm{WW}}_{\mathrm{S}}\,, \quad \mathcal{H}^{\mathrm{WW}} a^{\mathrm{WW}}_{\mathrm{L}} = \lambda_{\mathrm{L}} a^{\mathrm{WW}}_{\mathrm{L}}\,, \tag{39}$$

where $a^{\mathrm{WW}}_{\mathrm{S}}$ and $a^{\mathrm{WW}}_{\mathrm{L}}$ are eigenfunctions of the operator $\mathcal{H}^{\mathrm{WW}}$ for the eigenvalues

$$\lambda_{\mathrm{S}} = m_{\mathrm{S}} - \frac{\mathrm{i}}{2}\Gamma_{\mathrm{S}}, \quad \lambda_{\mathrm{L}} = m_{\mathrm{L}} - \frac{\mathrm{i}}{2}\Gamma_{\mathrm{L}}\,. \tag{40}$$

In the case of a CPT invariant system, (34), the relations (36) hold, which leads to the following form of the eigenvectors for $\mathcal{H}^{\mathrm{WW}}$ [6]:

$$|K_{\mathrm{S}}\rangle \equiv \frac{1}{p\left(1+\left|\frac{q}{p}\right|^2\right)^{\frac{1}{2}}}(p|K^0\rangle - q|\bar{K}^0\rangle) \tag{41}$$

and

$$|K_{\mathrm{L}}\rangle \equiv \frac{1}{p\left(1+\left|\frac{q}{p}\right|^2\right)^{\frac{1}{2}}}(p|K^0\rangle + q|\bar{K}^0\rangle)\,, \tag{42}$$

or, equivalently,

$$|K_{\mathrm{S}}\rangle = \rho^{\mathrm{WW}}\left(|K^0\rangle - r|\bar{K}^0\rangle\right), \tag{43}$$

$$|K_{\mathrm{L}}\rangle = \rho^{\mathrm{WW}}\left(|K^0\rangle + r|\bar{K}^0\rangle\right), \tag{44}$$

where

$$q \equiv \sqrt{\mathcal{H}^{\mathrm{WW}}_{\bar{K}^0K^0}}\,, \quad p \equiv \sqrt{\mathcal{H}^{\mathrm{WW}}_{K^0\bar{K}^0}}\,, \tag{45}$$

and

$$r = \frac{q}{p} = \sqrt{\frac{\mathcal{H}^{\mathrm{WW}}_{\bar{K}^0K^0}}{\mathcal{H}^{\mathrm{WW}}_{K^0\bar{K}^0}}}\,. \tag{46}$$

So within the WW approximation the physical states of the neutral kaons, K_{S} and K_{L}, are linear superpositions of K^0 and $\bar{K}^0$ and they decay exponentially evolving in time in $\mathfrak{H}_{||}$ (see (28) and (39)),

$$|K_{\mathrm{S}}(t)\rangle_{\parallel} = \mathrm{e}^{-\mathrm{i}\mathcal{H}^{\mathrm{WW}}t}|K_{\mathrm{S}}\rangle = \mathrm{e}^{-\mathrm{i}(m_{\mathrm{S}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{S}})t}|K_{\mathrm{S}}\rangle \in \mathfrak{H}_{||}\,, \tag{47}$$

$$|K_{\mathrm{L}}(t)\rangle_{\parallel} = \mathrm{e}^{-\mathrm{i}\mathcal{H}^{\mathrm{WW}}t}|K_{\mathrm{L}}\rangle = \mathrm{e}^{-(\mathrm{i}m_{\mathrm{L}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{L}})t}|K_{\mathrm{L}}\rangle \in \mathfrak{H}_{||}\,. \tag{48}$$

These last two relations and (43) and (44) enable one to determine the time evolution of the vectors $|K^0\rangle$ and $|\bar{K}^0\rangle$ in $\mathfrak{H}_{||}$. One finds, e.g., that within the WW approximation

$$\begin{aligned}|K^0(t)\rangle_{\parallel} &\simeq |K^0_{\mathrm{WW}}(t)\rangle_{\parallel} = \mathrm{e}^{-\mathrm{i}\mathcal{H}^{\mathrm{WW}}t}|K^0\rangle \\ &= \frac{1}{2}\left[\mathrm{e}^{-\mathrm{i}(m_{\mathrm{L}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{L}})t} + \mathrm{e}^{-\mathrm{i}(m_{\mathrm{S}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{S}})t}\right]|K^0\rangle \\ &\quad + \frac{r}{2}\left[\mathrm{e}^{-\mathrm{i}(m_{\mathrm{L}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{L}})t} - \mathrm{e}^{-\mathrm{i}(m_{\mathrm{S}}-\frac{\mathrm{i}}{2}\Gamma_{\mathrm{S}})t}\right]|\bar{K}^0\rangle\,.\end{aligned} \tag{49}$$

From the relations (41) and (42) follows another fundamental conclusion of the LOY theory about the properties of CPT invariant systems of neutral mesons. Namely the physical states $K_{\mathrm{L}}, K_{\mathrm{S}}$ have the form of (41) and (42) only if the condition (34) and thus the properties (36) hold. It is easy to calculate the scalar product of the vectors $|K_{\mathrm{S}}\rangle, |K_{\mathrm{L}}\rangle$, and one finds that

$$\langle K_{\mathrm{S}}|K_{\mathrm{L}}\rangle = \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} = (\langle K_{\mathrm{S}}|K_{\mathrm{L}}\rangle)^* = \langle K_{\mathrm{L}}|K_{\mathrm{S}}\rangle \neq 0\,. \tag{50}$$

This property means that within the LOY theory (i.e., within the WW approximation) the imaginary part, $\mathrm{Im}(\langle K_{\mathrm{S}}|K_{\mathrm{L}}\rangle)$, of the product $\langle K_{\mathrm{S}}|K_{\mathrm{L}}\rangle$ can be considered as the measure of a possible violation of the CPT symmetry: LOY theory states that the system considered is CPT invariant only if $\mathrm{Im}(\langle K_{\mathrm{S}}|K_{\mathrm{L}}\rangle) = 0$ [3,5,7–18].

3 Khalfin's effective Hamiltonian for the neutral kaon complex

In [6] the observation is made that the approximation $R(E) \simeq R(m)$ is not the best and leads to an indeterminate error in evaluating the amplitude $a(t)$, (17). It is obvious that using the more accurate estimation of $R(E)$ should yield a more accurate formula for $a(t)$. The suggestion is made in [6] that the more accurate approximation for $R(E)$ can be obtained expanding $R(E)$ into its Taylor series expansion around the point $E = m$. This idea gives

$$R(E) = R(m) + (E-m)\left.\frac{\mathrm{d}R(E)}{\mathrm{d}E}\right|_{E=m} + \frac{(E-m)^2}{2}\left.\frac{\mathrm{d}^2R(E)}{\mathrm{d}E^2}\right|_{E=m} + \dots \tag{51}$$

So the minimal improvement of the approximation $R(E) \simeq R(m)$ is the following one [6]:

$$R(E) \simeq R(m) + (E-m)\frac{\mathrm{d}R(m)}{\mathrm{d}m}, \tag{52}$$

where

$$\frac{\mathrm{d}R(m)}{\mathrm{d}m} \equiv \left.\frac{\mathrm{d}R(E)}{\mathrm{d}E}\right|_{E=m}. \tag{53}$$

Having this improved estimation of $R(E)$ and looking for a more accurate expression for the amplitude $a(t)$ the WW approximation defined in the previous section is modified in [6] by ignoring point 2 in the mentioned definition and replacing the approximation $R(E) \simeq R(m)$ used there by the relation (52). This procedure is Khalfin's improvement of the WW approximation.

Using (52) the denominator of the expression under the integral in formula (17) for the amplitude $a(t)$ takes the following form:

$$E - m - R(E) \simeq \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right) \times \left[E - m - \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} R(m)\right]. \tag{54}$$

Inserting (54) into (17) yields

$$a(t) \simeq \tilde{a}(t) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}E \left\{ \mathrm{e}^{-\mathrm{i}Et} \times \frac{1}{E - m - \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} R(m) + \mathrm{i}\varepsilon} \right\} \times \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} a(0), \quad (t > 0). \tag{55}$$

This expression for $a(t)$ replaces and improves the approximate formula (27) for $a(t) \simeq a^{\mathrm{WW}}(t)$.

Taking into account only the pole contribution into the value of the integral (55) leads to the result [6]

$$a(t) \simeq \tilde{a}(t) = \mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t} \tilde{a}(0), \quad (t > 0), \tag{56}$$

where

$$\tilde{a}(0) \stackrel{\mathrm{def}}{=} \mathbf{A} a(0) \equiv \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} a(0), \tag{57}$$

and

$$\tilde{\mathcal{H}} \equiv m \mathbb{I}_{\parallel} + \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} R(m) = \tilde{M} - \frac{\mathrm{i}}{2} \tilde{\Gamma}, \tag{58}$$

(where $\tilde{M} = \tilde{M}^{+}$ and $\tilde{\Gamma} = \tilde{\Gamma}^{+}$) denotes Khalfin's improved effective Hamiltonian acting in the subspace $\mathfrak{H}_{||}$.

Note that $\tilde{a}(t)$ solves the following Schrödinger-like equation, which is similar to (33):

$$\mathrm{i} \frac{\partial \tilde{a}(t)}{\partial t} = \tilde{\mathcal{H}} \tilde{a}(t). \tag{59}$$

This is the evolution equation for the subspace of states $\mathfrak{H}_{||}$ of neutral mesons. One should expect that the solutions $\tilde{a}(t)$ of this equation with the improved $\tilde{\mathcal{H}}$ will lead to a more accurate description of the real properties of neutral mesons than the solutions $a^{\mathrm{WW}}(t)$, (28), of (33).

We have

$$\mathbf{A} = \left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)^{-1} = D \begin{pmatrix} 1 - \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m}, & \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m} \\ \frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m} & ,1 - \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} \end{pmatrix}, \tag{60}$$

where

$$D = \left[\det\left(1 - \frac{\mathrm{d}R(m)}{\mathrm{d}m}\right)\right]^{-1} \equiv \left[1 - \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} - \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} + \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} - \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m} \frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m}\right]^{-1}. \tag{61}$$

Taking into account (60) one infers from (58) that the matrix elements of Khalfin's effective Hamiltonian, $\tilde{\mathcal{H}}$, have the following form:

$$\tilde{\mathcal{H}}_{\alpha\alpha} = m + D\left(R_{\alpha\alpha}(m) - R_{\alpha\alpha}(m)\frac{\mathrm{d}R_{\beta\beta}(m)}{\mathrm{d}m} + R_{\beta\alpha}(m)\frac{\mathrm{d}R_{\alpha\beta}(m)}{\mathrm{d}m}\right) \equiv \tilde{M}_{\alpha\alpha} - \frac{\mathrm{i}}{2}\tilde{\Gamma}_{\alpha\alpha}, \tag{62}$$

$$\tilde{\mathcal{H}}_{\alpha\beta} = D\left(R_{\alpha\beta}(m) - R_{\alpha\beta}(m)\frac{\mathrm{d}R_{\beta\beta}(m)}{\mathrm{d}m} + R_{\beta\beta}(m)\frac{\mathrm{d}R_{\alpha\beta}(m)}{\mathrm{d}m}\right) \equiv \tilde{M}_{\alpha\beta} - \frac{\mathrm{i}}{2}\tilde{\Gamma}_{\alpha\beta}, \tag{63}$$

where $\alpha \neq \beta$ and $\alpha, \beta = K^0, \bar{K}^0$. These two last formulae differ from those obtained in [6] for $\tilde{\mathcal{H}}_{\alpha\beta}$ and $\tilde{\mathcal{H}}_{\alpha\alpha}$. Strictly speaking the last components in (62) and (63) and the components corresponding to them in Khalfin's formulae for the matrix elements of $\tilde{\mathcal{H}}$ are different. What is more, the examination of the expressions for $\tilde{\mathcal{H}}_{\alpha\alpha}, \tilde{\mathcal{H}}_{\alpha\beta}$ given and discussed in [6] shows that the mentioned different components in Khalfin's formulae for $\tilde{\mathcal{H}}_{\alpha\alpha}, \tilde{\mathcal{H}}_{\alpha\beta}$ are wrong in the general case. For this reason the formulae for $\tilde{\mathcal{H}}_{\alpha\alpha}, \tilde{\mathcal{H}}_{\alpha\beta}$ used in [6] are incorrect. This means that one cannot be sure that all conclusions drawn in [6] and following from the analysis of the properties of $\tilde{\mathcal{H}}_{\alpha\alpha}, \tilde{\mathcal{H}}_{\alpha\beta}$ obtained there reflect real properties of the neutral meson complexes.

Making use of the expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$, then taking

$$x = \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} + \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} - \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} + \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m} \frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m}, \tag{64}$$

and assuming that this x fulfils the condition $|x| \ll 1$, the expression (61) for D can be approximated by the following formula:

$$D \simeq 1+x \tag{65}$$

$$= 1+\frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m}+\frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} - \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m}\frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m} + \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m}\frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m}. \tag{66}$$

Inserting (66) into (62) and (63) and keeping in these formulae only expressions of order up to that of type $R_{\alpha\beta}(m)R_{\alpha'\beta'}(m)$, $R_{\alpha\beta}\frac{\mathrm{d}R_{\alpha'\beta'}(m)}{\mathrm{d}m}$ (where $\alpha,\alpha',\beta,\beta' = K^0$, $\bar{K}^0$) one obtains

$$\tilde{\mathcal{H}}_{\alpha\alpha} \simeq m + R_{\alpha\alpha}(m) + R_{\beta\alpha}(m)\frac{\mathrm{d}R_{\alpha\beta}(m)}{\mathrm{d}m} + R_{\alpha\alpha}(m)\frac{\mathrm{d}R_{\alpha\alpha}(m)}{\mathrm{d}m}, \tag{67}$$

$$\tilde{\mathcal{H}}_{\alpha\beta} \simeq R_{\alpha\beta}(m) + R_{\beta\beta}(m)\frac{\mathrm{d}R_{\alpha\beta}(m)}{\mathrm{d}m} + R_{\alpha\beta}(m)\frac{\mathrm{d}R_{\alpha\alpha}(m)}{\mathrm{d}m}, \tag{68}$$

where $\alpha \neq \beta$.

Some general properties of the matrix elements $\tilde{\mathcal{H}}_{\alpha\beta}$ (see (62) and (63), and (67) and (68)) of $\tilde{\mathcal{H}}$ follow from the symmetry properties of the total Hamiltonian H. Assuming CPT invariance of the system containing neutral mesons, (6), one obtains the following relations:

$$R_{K^0K^0}(m) = R_{\bar{K}^0\bar{K}^0}(m), \qquad \frac{\mathrm{d}R_{K^0K^0}(m)}{\mathrm{d}m} = \frac{\mathrm{d}R_{\bar{K}^0\bar{K}^0}(m)}{\mathrm{d}m}, \tag{69}$$

which are analogous to (35).

If the system is CP invariant beside the relations (69) the following additional ones hold too:

$$R_{K^0\bar{K}^0}(m) = R_{\bar{K}^0K^0}(m), \qquad \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m} = \frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m}. \tag{70}$$

On the other hand if CP symmetry is violated, then the relations (70) are not valid, and one has

$$R_{K^0\bar{K}^0}(m) \neq R_{\bar{K}^0K^0}(m), \qquad \frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m} \neq \frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m}. \tag{71}$$

Finally, if the system is CPT invariant and CP symmetry is not conserved, then the relations (69) and (71) occur. This means, by (62), that $\tilde{\mathcal{H}}_{K^0K^0} \neq \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0}$; that is, that

$$\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0} \neq 0\,. \tag{72}$$

So it appears that the minimal improvement of the standard WW approximation leads to the conclusion that one of the fundamental results of the LOY theory for CPT invariant systems, i.e., the relation (36), cannot be considered as universally valid for real systems.

A conclusion analogous to (72) was also obtained in [6], but for the reasons mentioned after (62) and (63) the formula for $(\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0})$ obtained therein differs from that used in the next section and following from (62) and (63).

4 Some properties of Khalfin's improved effective Hamiltonian $\tilde{\mathcal{H}}$

In this section we will assume that CPT symmetry holds in the system under consideration. In such a case the relations (62) and (69)–(71) lead to the following expression for the difference of diagonal matrix elements (72) of Khalfin's effective Hamiltonian for the neutral mesons complex:

$$\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0} \stackrel{\mathrm{def}}{=} 2\tilde{h}_z = D\left(R_{K^0\bar{K}^0}(m)\frac{\mathrm{d}R_{\bar{K}^0K^0}(m)}{\mathrm{d}m} - R_{\bar{K}^0K^0}(m)\frac{\mathrm{d}R_{K^0\bar{K}^0}(m)}{\mathrm{d}m}\right) \neq 0\,. \tag{73}$$

Sometimes it is convenient to express this difference in terms of the matrix elements $\Sigma_{\alpha\beta}(m)$ instead of $R_{\alpha\beta}(m)$. Taking into account formula (20) for $R_{\alpha\beta}(m)$ and making use of the fact that

$$\frac{\mathrm{d}H^{\mathrm{w}}_{\alpha\beta}}{\mathrm{d}m} \equiv 0, \tag{74}$$

one finds

$$\frac{\mathrm{d}R_{\alpha\beta}(m)}{\mathrm{d}m} = -\frac{\mathrm{d}\Sigma_{\alpha\beta}(m)}{\mathrm{d}m}. \tag{75}$$

Inserting (20) and (75) into (73) one obtains

$$\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0} = D\left(-H^{\mathrm{w}}_{K^0\bar{K}^0}\frac{\mathrm{d}\Sigma_{\bar{K}^0K^0}(m)}{\mathrm{d}m} + H^{\mathrm{w}}_{\bar{K}^0K^0}\frac{\mathrm{d}\Sigma_{K^0\bar{K}^0}(m)}{\mathrm{d}m} + \Sigma_{K^0\bar{K}^0}(m)\frac{\mathrm{d}\Sigma_{\bar{K}^0K^0}(m)}{\mathrm{d}m} - \Sigma_{\bar{K}^0K^0}(m)\frac{\mathrm{d}\Sigma_{K^0\bar{K}^0}(m)}{\mathrm{d}m}\right), \tag{76}$$

where for D, see (61), due to the property (35), one has

$$D = \left[1+2\frac{\mathrm{d}\Sigma_{K^0K^0}(m)}{\mathrm{d}m} + \left(-\frac{\mathrm{d}\Sigma_{K^0K^0}(m)}{\mathrm{d}m}\right)^2 - \frac{\mathrm{d}\Sigma_{K^0\bar{K}^0}(m)}{\mathrm{d}m}\frac{\mathrm{d}\Sigma_{\bar{K}^0K^0}(m)}{\mathrm{d}m}\right]^{-1}. \tag{77}$$

Ignoring in (76) terms of the form $\left(\Sigma_{\alpha\beta}(m)\frac{\mathrm{d}\Sigma_{\beta\alpha}(m)}{\mathrm{d}m}\right)$, where $\alpha \neq \beta$, yields

$$\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0} \simeq D\Bigg(-H^{\mathrm{w}}_{K^0\bar{K}^0}\frac{\mathrm{d}\Sigma_{\bar{K}^0K^0}(m)}{\mathrm{d}m} + H^{\mathrm{w}}_{\bar{K}^0K^0}\frac{\mathrm{d}\Sigma_{K^0\bar{K}^0}(m)}{\mathrm{d}m}\Bigg) \neq 0\,. \quad (78)$$

Note that from this last relation an important conclusion follows: if $H^{\mathrm{w}}_{K^0\bar{K}^0} = 0, H^{\mathrm{w}}_{\bar{K}^0K^0} = (H^{\mathrm{w}}_{K^0\bar{K}^0})^* = 0$, then $\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0} \simeq 0$ to very good accuracy. So, if the first order $|\Delta S| = 2$ transitions, e.g., $K^0 \rightleftharpoons \bar{K}^0$, are forbidden for interactions, H^{w}, responsible for the decays of neutral mesons, then the difference of the diagonal matrix elements of the effective Hamiltonian more accurate than the LOY effective Hamiltonian, $\mathcal{H}^{\mathrm{WW}}$, equals zero. This means that a CPT invariance test based on the LOY theory relations (36) can be no longer considered as a CPT symmetry test but rather as a test of the existence of the interactions causing the first order $|\Delta S| = 2$ transitions (see also [19]).

The eigenvectors of $\tilde{\mathcal{H}}$ for the eigenvalues

$$\tilde{\mu}_{\mathrm{L(S)}} = \tilde{m}_{\mathrm{L(S)}} - \frac{\mathrm{i}}{2}\tilde{\Gamma}_{\mathrm{L(S)}} \quad (79)$$

have the form [6, 20]

$$|\tilde{K}_{\mathrm{L}}\rangle = \tilde{\rho}_{\mathrm{L}}\Big(|K^0\rangle - \alpha_{\mathrm{L}}|\bar{K}^0\rangle\Big) \quad (80)$$

and

$$|\tilde{K}_{\mathrm{S}}\rangle = \tilde{\rho}_{\mathrm{S}}\Big(|K^0\rangle - \alpha_{\mathrm{S}}|\bar{K}^0\rangle\Big), \quad (81)$$

where the parameters $\tilde{\rho}_{\mathrm{L}}, \tilde{\rho}_{\mathrm{S}}$ can be chosen as the real parameters;

$$\alpha_{\mathrm{L(S)}} = \frac{\tilde{h}_z - (+)\tilde{h}}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}}, \quad (82)$$

and the definition of $\tilde{h}_z$ is given by (73), and we have

$$\tilde{h} = \sqrt{(\tilde{h}_z)^2 + \tilde{\mathcal{H}}_{K^0\bar{K}^0}\tilde{\mathcal{H}}_{\bar{K}^0K^0}}\,. \quad (83)$$

Sometimes one uses the following expression for the vectors $|\tilde{K}_{\mathrm{L}}\rangle$ and $|\tilde{K}_{\mathrm{S}}\rangle$ [3, 5, 7–18, 20, 21]:

$$|\tilde{K}_{\mathrm{L(S)}}\rangle \equiv N_{\mathrm{L(S)}}[(1+\epsilon_{\mathrm{l(s)}})|K^0\rangle + (-1)(1-\epsilon_{\mathrm{l(s)}})|\bar{K}^0\rangle]. \quad (84)$$

This form of eigenvectors for the effective Hamiltonian is used in many papers when possible departures from the CP – or CPT – symmetry in the system considered are discussed. The following parameters are used to describe the scale of CP – and possible CPT – violation effects [3, 5, 7–18, 20]:

$$\epsilon \stackrel{\mathrm{def}}{=} \frac{1}{2}(\epsilon_{\mathrm{s}} + \epsilon_{\mathrm{l}})\,, \quad \delta \stackrel{\mathrm{def}}{=} \frac{1}{2}(\epsilon_{\mathrm{s}} - \epsilon_{\mathrm{l}}). \quad (85)$$

Within the LOY theory of the time evolution in the subspace of neutral kaons, ϵ describes violations of the CP symmetry and δ is considered as a CPT-violating parameter.

It seems to be interesting to compare the eigenvectors $|\tilde{K}_{\mathrm{L}}\rangle$, (80), and $|\tilde{K}_{\mathrm{S}}\rangle$, (81), for $\tilde{\mathcal{H}}$ with those corresponding to them (i.e, $|K_{\mathrm{S}}\rangle$, (43), and $|K_{\mathrm{L}}\rangle$, (44)), for $\mathcal{H}^{\mathrm{WW}}$. To achieve this goal one should rewrite the suitable expressions (82), so as to get a convenient form of $\alpha_{\mathrm{L(S)}}$ allowing one to express it by means of r, (46). After some algebra one can rewrite (82) as

$$\alpha_{\mathrm{L(S)}} = \tilde{g} - (+)\tilde{r}\sqrt{1 + \frac{\tilde{g}^2}{\tilde{r}^2}}, \quad (86)$$

where

$$\tilde{r} = \sqrt{\frac{\tilde{\mathcal{H}}_{\bar{K}^0K^0}}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}}} \quad (87)$$

and

$$\tilde{g} = \frac{\tilde{h}_z}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}}\,. \quad (88)$$

The conclusion that the more accurate approximation leads to a more realistic description of the properties of a physical system seems to be obvious. For this reason it seems that expressing the matrix elements of the more accurate $\tilde{\mathcal{H}}$ instead of $\mathcal{H}^{\mathrm{WW}}$ in terms of the parameters obtained from experiments is justified. So using the form (84) of the eigenvectors for $\tilde{\mathcal{H}}$, the matrix elements $\tilde{\mathcal{H}}_{\alpha\beta}$ of $\tilde{\mathcal{H}}$ can be expressed in terms of the observables ϵ_{L} and ϵ_{S} and $\tilde{\mu}_{\mathrm{L(S)}}$ (see, e.g. [14, 20]). By means of this method one finds, e.g., that

$$\tilde{g} = \frac{\tilde{h}_z}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}} \equiv \frac{1}{2}\frac{\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0}}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}} = \frac{2\delta}{(1+\epsilon_{\mathrm{l}})(1+\epsilon_{\mathrm{s}})}, \quad (89)$$

and

$$\tilde{r} = \sqrt{\frac{(1-\epsilon_{\mathrm{l}})(1-\epsilon_{\mathrm{s}})}{(1+\epsilon_{\mathrm{l}})(1+\epsilon_{\mathrm{s}})}}. \quad (90)$$

Experimentally measured values of the parameters $\epsilon_{\mathrm{l}}, \epsilon_{\mathrm{s}}$ are very small for neutral kaons. So, assuming

$$|\varepsilon_{\mathrm{l}}| \ll 1, \quad |\varepsilon_{\mathrm{s}}| \ll 1\,, \quad (91)$$

one finds that

$$|\tilde{g}| \ll 1 \quad \text{and} \quad |\tilde{r}| \simeq 1\,, \quad (92)$$

and thus

$$\left|\frac{\tilde{g}}{\tilde{r}}\right| \ll 1\,. \quad (93)$$

Therefore to a very good approximation

$$\sqrt{1 + \frac{\tilde{g}^2}{\tilde{r}^2}} \simeq 1 + \frac{1}{2}\frac{\tilde{g}^2}{\tilde{r}^2}\,. \quad (94)$$

So, the relation (86) can be approximated by the following one:

$$\alpha_{\mathrm{L(S)}} \simeq \tilde{g} - (+)\tilde{r}\left(1 + \frac{\tilde{g}^2}{2\tilde{r}^2}\right), \quad (95)$$

or, taking into account (93), by

$$\alpha_{\rm L} \simeq -\tilde{r} + \tilde{g}, \quad \text{and} \quad \alpha_{\rm S} \simeq \tilde{r} + \tilde{g}\,. \tag{96}$$

Inserting (96) into (80) and (81) results in the following expressions for $|\tilde{K}_{\rm L}\rangle and |\tilde{K}_{\rm S}\rangle$:

$$|\tilde{K}_{\rm L}\rangle = \tilde{\rho}_{\rm L}\left(|K^0\rangle + \tilde{r}|\bar{K}^0\rangle\right) - \tilde{g}\tilde{\rho}_{\rm L}|\bar{K}^0\rangle \tag{97}$$

and

$$|\tilde{K}_{\rm S}\rangle = \tilde{\rho}_{\rm S}\left(|K^0\rangle - \tilde{r}|\bar{K}^0\rangle\right) - \tilde{g}\tilde{\rho}_{\rm S}|\bar{K}^0\rangle\,. \tag{98}$$

Next, let us note that

$$(\tilde{r})^2 = \frac{\tilde{\mathcal{H}}_{\bar{K}^0K^0}}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}} \equiv \frac{\mathcal{H}^{\rm WW}_{\bar{K}^0K^0} + R_{K^0K^0}(m)\frac{{\rm d}R_{\bar{K}^0K^0}(m)}{{\rm d}m} + R_{\bar{K}^0K^0}(m)\frac{{\rm d}R_{\bar{K}^0\bar{K}^0}(m)}{{\rm d}m}}{\mathcal{H}^{\rm WW}_{K^0\bar{K}^0} + R_{\bar{K}^0\bar{K}^0}(m)\frac{{\rm d}R_{K^0\bar{K}^0}(m)}{{\rm d}m} + R_{K^0\bar{K}^0}(m)\frac{{\rm d}R_{K^0K^0}(m)}{{\rm d}m}} \tag{99}$$

(because $\mathcal{H}^{\rm WW}_{K^0\bar{K}^0} \equiv R_{K^0\bar{K}^0}(m)$ and $\mathcal{H}^{\rm WW}_{\bar{K}^0K^0} \equiv R_{\bar{K}^0K^0}(m)$ – see (31)). This means that neglecting terms of the type $R_{\alpha\beta}(m)\frac{{\rm d}R_{\alpha'\beta'}(m)}{{\rm d}m}$, one finds

$$\frac{\tilde{\mathcal{H}}_{\bar{K}^0K^0}}{\tilde{\mathcal{H}}_{K^0\bar{K}^0}} \simeq \frac{\mathcal{H}^{\rm WW}_{\bar{K}^0K^0}}{\mathcal{H}^{\rm WW}_{K^0\bar{K}^0}}\,, \tag{100}$$

which enables one to draw the conclusion that the difference between the expressions (87) and (46) is almost negligibly small, i.e., that

$$\tilde{r} \simeq r\,, \tag{101}$$

to the very good approximation. So, in our analysis expressions (96) can be replaced by the following one:

$$\alpha_{\rm S(L)} \simeq +(-)r + \tilde{g}\,. \tag{102}$$

The conclusion following from this property is that the formulae (97) and (98) for the eigenvectors $|\tilde{K}_{\rm L}\rangle$ and $|\tilde{K}_{\rm S}\rangle$ of $\mathcal{H}$ can be rewritten using the eigenvectors $|K_{\rm L}\rangle$ and $|K_{\rm S}\rangle$, (43) and (44), of $\mathcal{H}^{\rm WW}$ as follows:

$$|\tilde{K}_{\rm L}\rangle = \frac{\tilde{\rho}_{\rm L}}{\rho^{\rm WW}}|K_{\rm L}\rangle - \tilde{g}\tilde{\rho}_{\rm L}|\bar{K}^0\rangle \tag{103}$$

and

$$|\tilde{K}_{\rm S}\rangle = \frac{\tilde{\rho}_{\rm S}}{\rho^{\rm WW}}|K_{\rm S}\rangle - \tilde{g}\tilde{\rho}_{\rm S}|\bar{K}^0\rangle\,. \tag{104}$$

To complete our analysis we should remove $|\bar{K}^0\rangle$ from (103) and (104). Expressing $|\bar{K}^0\rangle$ by $|K_{\rm S}\rangle$, (43), and $|K_{\rm L}\rangle$, (44), and then inserting the formula obtained for $|\bar{K}^0\rangle$ into (103) and (104) yields

$$|\tilde{K}_{\rm S}\rangle = \frac{\tilde{\rho}_{\rm S}}{\rho^{\rm WW}}\left[\left(1 + \frac{\tilde{g}}{2r}\right)|K_{\rm S}\rangle - \frac{\tilde{g}}{2r}|K_{\rm L}\rangle\right] \tag{105}$$

and

$$|\tilde{K}_{\rm L}\rangle = \frac{\tilde{\rho}_{\rm L}}{\rho^{\rm WW}}\left[\left(1 - \frac{\tilde{g}}{2r}\right)|K_{\rm L}\rangle + \frac{\tilde{g}}{2r}|K_{\rm S}\rangle\right]. \tag{106}$$

Similarly, starting from the relations (97) and (98) one can express $|K^0\rangle$ and $|\bar{K}^0\rangle$ by means of the eigenvectors $|\tilde{K}_{\rm L}\rangle$ and $|\tilde{K}_{\rm S}\rangle$ for $\tilde{\mathcal{H}}$. Next inserting the expressions obtained in this way for $|K^0\rangle$ and $|\bar{K}^0\rangle$ into the formulae for $|K_{\rm S}\rangle$, (43), and $|K_{\rm L}\rangle$, (44), and using property (101), one finds that

$$|K_{\rm S}\rangle \simeq \rho^{\rm WW}\left[\frac{1}{\tilde{\rho}_{\rm S}}\left(1 - \frac{\tilde{g}}{2r}\right)|\tilde{K}_{\rm S}\rangle + \frac{1}{\tilde{\rho}_{\rm L}}\frac{\tilde{g}}{2r}|\tilde{K}_{\rm L}\rangle\right], \tag{107}$$

and

$$|K_{\rm L}\rangle \simeq \rho^{\rm WW}\left[\frac{1}{\tilde{\rho}_{\rm L}}\left(1 + \frac{\tilde{g}}{2r}\right)|\tilde{K}_{\rm L}\rangle - \frac{1}{\tilde{\rho}_{\rm S}}\frac{\tilde{g}}{2r}|\tilde{K}_{\rm S}\rangle\right]. \tag{108}$$

All the above considerations are carried out within the assumption that the system containing neutral kaons is CPT invariant. This means that contrary to one of the fundamental results of the LOY theory (see (50)) the scalar product of the eigenvectors $|\tilde{K}_{\rm S}\rangle$ and $|\tilde{K}_{\rm L}\rangle$ for Khalfin's improved effective Hamiltonian $\tilde{\mathcal{H}}$ cannot be real in the CPT invariant system. Indeed from (105) and (106) it follows that

$$\begin{aligned}\langle\tilde{K}_{\rm S}|\tilde{K}_{\rm L}\rangle &= \frac{\tilde{\rho}_{\rm S}\tilde{\rho}_{\rm L}}{(\rho^{\rm WW})^2}\left[\left(1 - 2{\rm i}\,{\rm Im}(\frac{\tilde{g}}{2r}) - \frac{|\tilde{g}|^2}{2|r|^2}\right)\langle K_{\rm S}|K_{\rm L}\rangle\right.\\ &\quad\left. + 2{\rm i}\,{\rm Im}\left(\frac{\tilde{g}}{2r}\right) + \frac{|\tilde{g}|^2}{2|r|^2}\right]\\ &\neq (\langle\tilde{K}_{\rm S}|\tilde{K}_{\rm L}\rangle)^* = \langle\tilde{K}_{\rm L}|\tilde{K}_{\rm S}\rangle\,,\end{aligned} \tag{109}$$

where ${\rm Im}(z)$ denotes the imaginary part of the complex number z. (Note that according to our assumptions, the parameters $\tilde{\rho}_{\rm S}$, $\tilde{\rho}_{\rm L}$ and $\rho^{\rm WW}$ are real numbers and the product $\langle K_{\rm S}|K_{\rm L}\rangle$ is real too – see (50).)

Ignoring in (109) terms of order $\frac{|\tilde{g}|^2}{|r|^2}$ gives

$$\begin{aligned}\langle\tilde{K}_{\rm S}|\tilde{K}_{\rm L}\rangle &\simeq \frac{\tilde{\rho}_{\rm S}\tilde{\rho}_{\rm L}}{(\rho^{\rm WW})^2}\left[\left(1 - 2{\rm i}\,{\rm Im}(\frac{\tilde{g}}{2r})\right)\langle K_{\rm S}|K_{\rm L}\rangle\right.\\ &\quad\left. + 2{\rm i}\,{\rm Im}\left(\frac{\tilde{g}}{2r}\right)\right]\end{aligned} \tag{110}$$

to an accuracy sufficient for our analysis. Note that these last two relations are in perfect agreement with the result obtained in [21], where the general proof that the scalar product of eigenvectors for the exact effective Hamiltonian corresponding to short and long living superpositions of K^0 and $\bar{K}^0$ mesons cannot be real in a CPT invariant system is given.

5 Final remarks

At the beginning of this section we should explain why the final formulae for the matrix elements of the improved effective Hamiltonian $\tilde{\mathcal{H}}$ derived in Sect. 3 differ a little from those obtained by Khalfin in his paper [6]. This is because in [6] some additional assumptions were used to simplify the calculations. A detailed analysis of these assumptions and recent experimental data suggest that some of them cannot be considered as universally valid for neutral meson complex. In deriving our formulae, i.e. (62) and (63) (and (67) and (68)) for $\tilde{\mathcal{H}}_{\alpha\beta}$, only the general Khalfin assumptions having the form (52) and (54) have been used. The general form of $\tilde{\mathcal{H}}$ obtained in Sect. 3 and given by formula (58) and the formula for $\tilde{\mathcal{H}}$ derived in [6] are identical. On the other hand our formulae (62) and (63) for $\tilde{\mathcal{H}}_{\alpha\beta}$ seem to be more general and more accurate than those obtained in [6].

Note that if CP symmetry is conserved, then $H^{\mathrm{w}}_{K^0\bar{K}^0} = H^{\mathrm{w}}_{\bar{K}^0K^0}$ and the relations (70) are also valid. Then from (73) it follows that $(\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0}) = 0$ in a CP invariant system. This means that in such a case the picture of a neutral meson system obtained within the use of Khalfin's improved effective Hamiltonian, $\tilde{\mathcal{H}}$, is the same as that obtained using the LOY effective hamiltonian $\tilde{\mathcal{H}}^{\mathrm{WW}}$.

Let us now focus on the case of violated CP and conserved CPT symmetries. The most important conclusion following from Sects. 2–4 is that the minimal improvement of the WW approximation leads to such properties of the new effective Hamiltonian for the neutral kaon complex as a contradiction of standard predictions of the LOY theory for systems in which CP symmetry is violated. The relations (72), (73) and (78) are examples of such properties. These relations state that the standard result of the LOY theory, that diagonal matrix elements of the effective Hamiltonian governing the time evolution of neutral kaons are equal if the system containing these neutral K mesons is CPT invariant, cannot be considered as a true property for the real systems. Another example is the relation (109). This result obtained for the eigenvectors of Khalfin's improved effective Hamiltonian shakes another standard prediction of the LOY theory, i.e., the property (50), that the scalar product of eigenvectors for the effective Hamiltonian should be a real number for a system preserving CPT symmetry. All these corrections to the corresponding LOY results are very small. They are of order of the parameter $\tilde{g}$ defined by the relations (88) and (73). Nevertheless they all have a nonzero value. This means that the LOY theory interpretation of experimentally measured parameters for neutral meson complexes may not properly reflect all real properties of such complexes. For example, the properties (72), (73) and (78) mean by (85) and (89) that there must be $\epsilon_{\mathrm{l}} \neq \epsilon_{\mathrm{s}}$ when CPT symmetry holds and CP is violated (see also [22, 23]). Of course this conclusion contradicts the standard predictions of the LOY theory.

One more observation is in agreement with the last conclusions. Namely assuming that the picture of the physical system given by the parameters calculated within the more accurate approximation is more realistic than that following from the less accurate one, from (107) and (108) we can draw the following conclusion: superpositions of the states $|K^0\rangle$ and $|\bar{K}^0\rangle$ of type $|K_{\mathrm{L}}\rangle, |K_{\mathrm{S}}\rangle$, having the form (43) and (44), with expansion coefficients r, see (46), calculated within the WW approximation, cannot be considered as the real physical states. The relations (107) and (108) show that the vectors $|K_{\mathrm{L}}\rangle$, (43), and $|K_{\mathrm{S}}\rangle$, (44), are linear combinations of the eigenvectors $|\tilde{K}_{\mathrm{L}}\rangle and|\tilde{K}_{\mathrm{S}}\rangle$ for the $\tilde{\mathcal{H}}$ that is more accurate than $\tilde{\mathcal{H}}^{\mathrm{WW}}$. It seems that rather the vectors $|\tilde{K}_{\mathrm{L}}\rangle$ and $|\tilde{K}_{\mathrm{S}}\rangle$ as the eigenvectors of the more accurate effective Hamiltonian should claim to represent the physical states of the neutral kaons.

It seems to be interesting that the relation (78) confirms the result obtained in [19] within the use of a different formalism than that leading to the result (78) in Sect. 3. This result and the result discussed in [19] suggest that tests for neutral meson complexes based on the measurement of the difference of the diagonal matrix elements of the effective Hamiltonian for such a complex cannot be considered as CPT invariance tests. From (78) it follows that this difference equals zero in a CPT invariant system if $\langle K^0|H^{\mathrm{w}}|\bar{K}^0\rangle = 0$ and does not equal zero if $\langle K^0|H^{\mathrm{w}}|\bar{K}^0\rangle \neq 0$. This means that this difference cannot be equal to zero if the first order $|\Delta S| = 2$ transitions $K^0 \rightleftharpoons \bar{K}^0$ take place in the system considered. So the tests mentioned should rather be considered as tests for the existence of interactions causing the first order $|\Delta S| = 2$ transitions $K^0 \rightleftharpoons \bar{K}^0$ in the system.

Note also that the properties (73) and (109) of Khalfin's effective Hamiltonian are in perfect agreement with the analogous rigorous results obtained in [24] and [21] without the use of any approximations for the exact effective Hamiltonian.

It is also interesting to compare the solutions (28) of the LOY evolution equation (33) for the subspace $\mathfrak{H}_{\|}$ with the solutions (56) of the evolution equation (59) containing Khalfin's improved effective Hamiltonian $\tilde{\mathcal{H}}$. We already have a solution $|K^0_{\mathrm{WW}}(t)\rangle_{\|}$, (49), of (33). Let us find an analogous solution $|\tilde{K}^0(t)\rangle_{\|}$ of (59). From (56) one finds

$$|K^0(t)\rangle_{\|} \simeq |\tilde{K}^0(t)\rangle_{\|} = \mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t}|\tilde{K}^0(0)\rangle_{\|}\,, \quad (t>0)\,, \qquad (111)$$

where, according to (57),

$$|\tilde{K}^0(0)\rangle_{\|} = \mathbf{A}|K^0\rangle \equiv a_{11}|K^0\rangle + a_{21}|\bar{K}^0\rangle\,, \qquad (112)$$

and the a_{jk} $(j,k=1,2)$ are matrix elements of $\mathbf{A}$ (see (57) and (60)). Thus

$$\mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t}|\tilde{K}^0(0)\rangle_{\|} \equiv a_{11}\mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t}|K^0\rangle + a_{21}\mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t}|\bar{K}^0\rangle\,. \qquad (113)$$

Next using the relations (80) and (81) one can express the vectors $|K^0\rangle$ and $|\bar{K}^0\rangle$ by means of the eigenvectors $|\tilde{K}_{\mathrm{L}}\rangle$ and $|\tilde{K}_{\mathrm{S}}\rangle$ for $\tilde{\mathcal{H}}$. The result of the action of $\mathrm{e}^{-\mathrm{i}\tilde{\mathcal{H}}t}$ onto $|\tilde{K}_{\mathrm{L}}\rangle$ and $|\tilde{K}_{\mathrm{S}}\rangle$ can easily be found. Having this result one can return to the base vectors $|K^0\rangle$ and $|\bar{K}^0\rangle$, which yields

$$|\tilde{K}^0(t)\rangle_{\|} = \frac{1}{\alpha_{\mathrm{S}} - \alpha_{\mathrm{L}}}\Big[(a_{11}\alpha_{\mathrm{S}} + a_{21})\mathrm{e}^{-\mathrm{i}(\tilde{m}_{\mathrm{L}} - \frac{\mathrm{i}}{2}\tilde{\Gamma}_{\mathrm{L}})t} - (a_{11}\alpha_{\mathrm{L}} + a_{21})\mathrm{e}^{-\mathrm{i}(\tilde{m}_{\mathrm{S}} - \frac{\mathrm{i}}{2}\tilde{\Gamma}_{\mathrm{S}})t}\Big]|K^0\rangle$$

$$- \frac{1}{\alpha_S - \alpha_L}\Big[(a_{11}\alpha_S + a_{21})\alpha_L e^{-i(\tilde{m}_L - \frac{i}{2}\tilde{\Gamma}_L)t} - (a_{11}\alpha_L + a_{21})\alpha_S e^{-i(\tilde{m}_S - \frac{i}{2}\tilde{\Gamma}_S)t}\Big]|\bar{K}^0\rangle\,. \quad (114)$$

Note that this last expression is not equal to the analogous formula obtained in [6]. The cause of this is described in Sect. 4 after formulae (62) and (63). Nevertheless the general conclusions following from (114) and from the corresponding Khalfin formula mentioned are similar. Namely from (49) and from (114) (as well as from the analogous formula obtained in [6]), it follows that

$$|\langle K^0|K^0_{WW}(t)\rangle_{||}|^2 \neq |\langle K^0|\tilde{K}^0(t)\rangle_{||}|^2,$$
$$|\langle \bar{K}^0|K^0_{WW}(t)\rangle_{||}|^2 \neq |\langle \bar{K}^0|\tilde{K}^0(t)\rangle_{||}|^2.$$

Thus within Khalfin's improved theory of neutral mesons, formulae describing strangeness (particle–antiparticle) oscillations lead to some corrections to the corresponding standard predictions of the LOY theory. This effect was called in [6] "a new CP violation effect". So if more accurate tests of the mentioned oscillations detect some departures from the predictions obtained within the use of the probabilities $|\langle K^0|K^0_{WW}(t)\rangle_{||}|^2$ and $|\langle \bar{K}^0|K^0_{WW}(t)\rangle_{||}|^2$, then such an effect should be considered as a very probable confirmation of the improved Khalfin's theory of neutral mesons.

The complementary conclusion to the above one is the observation following from the results described in Sect. 4: more accurate measurements of the difference $(\tilde{\mathcal{H}}_{K^0K^0} - \tilde{\mathcal{H}}_{\bar{K}^0\bar{K}^0})$, that is of the parameters ϵ_L, ϵ_S and δ, (85) (see (89) and (73)), should also make it possible to see the difference between the standard predictions of the LOY theory and the theory based on the more accurate effective Hamiltonian derived by Khalfin.

The investigations of the above mentioned "new CP violation effect" and related problems by means of the other and more general method than that used in [6] were continued in [25–27]. The expected form of the probabilities $|\langle K^0|\tilde{K}^0(t)\rangle_{||}|^2$ and $|\langle \bar{K}^0|\tilde{K}^0(t)\rangle_{||}|^2$ is given in [26, 27].

One last comment. In [28] (see also [20]) a new approach to describing the time evolution in neutral kaon complex was proposed. This approach is based on the exact evolution equation for a subspace of states, $\mathfrak{H}_{||}$, describing neutral mesons, sometimes called the Królikowski–Rzeuwski (KR) equation for the distinguished component of a state vector [29, 30]. Using the KR equation the approximate effective Hamiltonian for the $H_{||}$ neutral meson complex was derived. The approximation used there has the advantage over the WW and LOY approximations that all steps leading to the final formulae for the approximate $H_{||}$ are well defined. $H_{||}$ thus obtained differs from $\mathcal{H}^{WW}$, and its matrix elements have a form close to that of $\tilde{\mathcal{H}}$. In detail: the property of $\tilde{\mathcal{H}}$ of type (73) occurs also for the difference of the diagonal matrix elements of the $H_{||}$. Replacing D in the formula (78) by $D \simeq \mathbb{I}$ (which for some purposes is a sufficient approximation), the relation (78) becomes identical with the analogous relation derived in [19] for the $H_{||}$. The property (109) of the scalar product of the eigenvectors for $\tilde{\mathcal{H}}$ occurs also for the eigenvectors of the approximate $H_{||}$ obtained within the use of the KR equation.

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